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Oscillation for System of Delay Difference Equations*

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Consider the system of delay difference equations

$$x_i(t) - x_i(t-\sigma) + \sum_{k=1}^{l}\sum_{j=1}^{n} p_{ijk}x_j(t-\tau_k) = 0, \qquad (*)$$

where $p_{ijk} \in (-\infty, \infty)$, σ and $\tau_k \in (0, \infty)$, $i, j = 1, 2, \ldots, n, k = 1, 2, \ldots, l$. Sufficient conditions are obtained for all solutions of the system $(*)$ to be oscillatory.

Key Words: oscillation; delay difference system; difference inequality

1. INTRODUCTION

Consider the system of delay difference equations

$$x_i(t) - x_i(t-\sigma) + \sum_{k=1}^{l}\sum_{j=1}^{n} p_{ijk}x_j(t-\tau_k) = 0, \qquad i = 1, 2, \ldots, n, \qquad (1)$$

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where

$$\sigma, \tau_k \in (0, \infty) \text{ and } p_{ijk} \in R, \quad \text{for } i, j = 1, 2, \ldots, n \text{ and } k = 1, 2, \ldots, l. \tag{2}$$

Our aim of this paper is to study the oscillatory behavior of solutions of (1). When $l = n = 1$, the system (1) reduces to a scalar delay difference equation whose oscillatory behavior caused by delays has been investigated in [3] and [4]. For the oscillation theory of delay differential-difference equations, we refer to the monographs by Gopalsamy [1] and Gyori and Ladas [2].

Let $\gamma = \max\{\sigma, \tau_k, \quad k = 1, 2, \ldots, l\}$. By a solution $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ of (1) we mean a continuous function $x \in C[[t_0 - \gamma, \infty), R^n]$ which satisfies (1) for all $t \geq t_0$. A solution $x(t)$ of (1) is said to be oscillatory if at least one of its components $x_i(t)$ has arbitrarily large zeros. Otherwise the solution is called nonoscillatory. In particular, a solution of scalar difference equation is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

In what follows, for the sake of convenience, when we write a functional inequality without specifying its domain of validity, we will assume it holds for all large t.

2. SOME LEMMAS

In this section we present some lemmas which will be utilized in the proofs of the main results in Section 3.

LEMMA 1. *Assume that* (2) *hold. If* (1) *has a nonoscillatory solution* $x(t)$, *then there are numbers* $\delta_i \in \{-1, 1\}$, $i = 1, 2, \ldots, n$ *such that*

$$y_i(t) - y_i(t - \sigma) + \sum_{k=1}^{l} \sum_{j=1}^{n} \bar{p}_{ijk} y_j(t - \tau_k) = 0, \qquad i = 1, 2, \ldots, n, \tag{3}$$

where

$$\bar{p}_{ijk} = \frac{\delta_i}{\delta_j} p_{ijk}, \qquad i, j = 1, 2, \ldots, n, \quad \text{and} \quad k = 1, 2, \ldots, l, \tag{4}$$

has a nonoscillatory solution $y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$ *with eventually positive components* $y_i(t)$, $i = 1, 2, \ldots, n$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1) with eventually positive or negative components. Then there exists $T \geq t_0$ such that

$x_i(t) \neq 0$ for $t \geq T$ and $i = 1, 2, \ldots, n$. Set $\delta_i = \operatorname{sign}[x_i(t)]$, $i = 1, 2, \ldots, n$. It is easy to see that

$$y(t) = \left[\delta_1 x_1(t), \delta_2 x_2(t), \ldots, \delta_n x_n(t)\right]^T$$

satisfies (3) and $y_i(t) = \delta_i x_i(t) > 0$ for $t \geq T$ and $i = 1, 2, \ldots, n$. The proof of Lemma 1 is complete. ∎

Remark 1. Clearly, from (4), for $i, j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, l$,

$$|\bar{p}_{ijk}| = |p_{ijk}| \quad \text{and} \quad \bar{p}_{iik} = p_{iik}.$$

LEMMA 2. *Assume that* $p_k, \tau_k \in (0, \infty)$, $k = 1, 2, \ldots, l$, $\sigma \in (0, \infty)$, *and*

$$\sigma < \tau = \max_{1 \leq k \leq l} \{\tau_k\}. \tag{5}$$

If the difference inequality

$$u(t) - u(t - \sigma) + \sum_{k=1}^{l} p_k u(t - \tau_k) \leq 0 \tag{6}$$

has an eventually positive solution $u(t)$, *then difference equation*

$$v(t) - v(t - \sigma) + \sum_{k=1}^{l} p_k v(t - \tau_k) = 0 \tag{7}$$

has also an eventually positive solution $v(t)$ *and*

$$v(t) \leq z(t),$$

where $z(t) = \int_{t-\sigma}^{t} u(s)\, ds$ *and* $\lim_{t \to \infty} z(t) = 0$.

Proof. Suppose that (6) has an eventually positive solution $u(t) > 0$ for $t \geq T > 0$. By integrating both sides of (6) on $[t - \sigma, t]$, $t \geq T$, we obtain

$$\int_{t-\sigma}^{t} u(s)\, ds - \int_{t-\sigma}^{t} u(s - \sigma)\, ds + \sum_{k=1}^{l} p_k \int_{t-\sigma}^{t} u(s - \tau_k)\, ds \leq 0 \quad \text{for all } t \geq T + \tau. \tag{8}$$

Set

$$z(t) = \int_{t-\sigma}^{t} u(s)\, ds \quad \text{for every} \quad t \geq T + \tau.$$

Thus $z(t) > 0$ and $z'(t) = u(t) - u(t-\sigma) = -\sum_{k=1}^{l} p_k u(t-\tau_k) < 0$. Hence $z(t)$ is decreasing and

$$\lim_{t \to \infty} z(t) = L \in [0, \infty). \tag{9}$$

By using (8), it is easy to prove $L = 0$ and

$$z(t) - z(t-\sigma) + \sum_{k=1}^{l} p_k z(t-\tau_k) \le 0 \quad \text{for } t \ge T + \tau.$$

Further, we have

$$z(t+m\sigma) - z(t+(m-1)\sigma) + \sum_{k=1}^{l} p_k z(t+m\sigma-\tau_k) \le 0$$
$$\text{for } t \ge T+\tau \text{ and } m = 1, 2, \ldots. \tag{10}$$

Summing (10) for m from 1 to N, we obtain

$$z(t+N\sigma) + \sum_{m=1}^{N} \sum_{k=1}^{l} p_k z(t+m\sigma-\tau_k) \le z(t), \quad \text{for all } t \ge T+\tau. \tag{11}$$

By taking limits as $N \to \infty$ in the left of (11), in view of (9), we find

$$\sum_{m=1}^{\infty} \sum_{k=1}^{l} p_k z(t+m\sigma-\tau_k) \le z(t) \quad \text{for all } t \ge T+\tau. \tag{12}$$

Now we consider the set W of all nonnegative continuous functions w satisfying the following conditions:

$$W = \{ w \in C([T+\tau, \infty), [0,\infty)) | 0 \le w(t) \le z(t) \text{ for every } t \ge T+\tau \}$$

and a mapping F on W as follows:

$$(Fw)(t) = \begin{cases} \displaystyle\sum_{m=1}^{\infty} \sum_{k=1}^{l} p_k w(t+m\sigma-\tau_k), \\ \qquad \text{for } t \ge T+2\tau-\sigma, \\ (Fw)(T+2\tau-\sigma) + z(t) - z(t+2\tau-\sigma), \\ \qquad \text{for } T+\tau \le t \le t+2\tau-\sigma. \end{cases} \tag{13}$$

First, we prove that mapping (13) is continuous. As $\lim_{t \to \infty} z(t) = 0$, we know that for any $\varepsilon > 0$ there exists $T_1 \ge T+\tau$ such that $z(t) < \varepsilon$ for all

$t \geq T_1$. We choose positive integer $N \geq T_1/\sigma$. Then, from (12), for any $m_2 > m_1 \geq N$ and all $t \geq T + \tau$, we obtain

$$\begin{aligned}\sum_{m=m_1+1}^{m_2} \sum_{k=1}^{l} p_k w(t + m\sigma - \tau_k) &\leq \sum_{m=m_1+1}^{\infty} \sum_{k=1}^{l} p_k z(t + m\sigma - \tau_k) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{l} p_k z\big(t + (m_1 + m)\sigma - \tau_k\big) \\ &< z(t + m_1\sigma) < \varepsilon,\end{aligned}$$

which implies that the series $\sum_{m=1}^{\infty} \sum_{k=1}^{l} p_k w(t + m\sigma - \tau_k)$ of functions defined on $[T + \tau, \infty)$ converges uniformly on $[T + \tau, \infty)$. Thus from (13), $Fw\colon [T + \tau, \infty) \to [0, \infty)$ is continuous.

Next, (13) defines an increasing mapping $F\colon W \to W$. The increasing character of F is considered with respect to the usual pointwise ordering in W, that is, for any w_1 and $w_2 \in W$ with $w_1(t) \leq w_2(t)$ implies $(Fw_1)(t) \leq (Fw_2)(t)$. Note also by (12) that $(Fw)(t) \leq z(t)$ for all $t \geq T + \tau$.

Consider the decreasing sequence $\{v_n\}_{n=0}^{\infty}$ of functions in W defined by $v_0(t) = z(t)$ and $v_m(t) = (Fv_{m-1})(t)$, $m = 1, 2, \dots,$ and set

$$v(t) = \lim_{m \to \infty} v_m(t) \quad \text{pointwise on } [T + \tau, \infty). \tag{14}$$

From (13) and (14), we can apply convergence theorem to obtain $v(t) = (Fv)(t)$, that is,

$$v(t) = \sum_{m=1}^{\infty} \sum_{k=1}^{l} p_k v(t + m\sigma - \tau_k) \quad \text{for every } t \geq T + 2\tau - \sigma. \tag{15}$$

Since $v_m(t)$ converges uniformly on $[T + \tau, \infty)$, it follows from (15) that $v(t)$ is continuous on $[t + \tau, \infty)$ and

$$\begin{aligned}v(t) - v(t - \sigma) &= \sum_{m=1}^{\infty} \sum_{k=1}^{l} p_k v(t - m\sigma - \tau_k) \\ &\quad - \sum_{m=1}^{\infty} \sum_{k=1}^{l} p_k v\big(t + (m - 1)\sigma - \tau_k\big) \\ &= -\sum_{k=1}^{l} p_k v(t - \tau_k),\end{aligned} \tag{16}$$

which means that $v(t)$ is a solution on $[T + \tau, \infty)$ of (7) with $v(t) \leq z(t) = \int_{t-\sigma}^{t} u(s)\,ds$.

It remains to prove that v is positive on $[T + \tau, \infty)$. For $T + \tau \le t < T + 2\tau - \sigma$, from (13) we have $0 < z(t) - z(t - 2\tau - \sigma) < v(t)$. Hence $v(t) > 0$ on $[T + \tau, T + 2\tau - \sigma)$. Let $t^* = \inf\{t \ge T + 2\tau - \sigma | v(t) = 0\}$. We will prove $t^* = \infty$. Otherwise, $t^* \in [T + 2\tau - \sigma, \infty)$. So $v(t) > 0$ for $T + \tau < t < t^*$ and $v(t^*) = 0$. But using (15), we have

$$v(t^*) = \sum_{m=1}^{\infty} \sum_{k=1}^{l} p_k v(t^* + m\sigma - \tau_k) \ge \sum_{k=1}^{l} p_k v(t^* + \sigma - \tau_k) > 0,$$

which is a contradiction. This contradiction implies $t^* = \infty$. The proof of Lemma 2 is complete. ∎

3. A COMPARISON THEOREM FOR OSCILLATION

In this section we will establish some sufficient conditions for the oscillation of system (1). First, we give the following comparison theorem.

THEOREM 1. *Let*

$$q_k = \sum_{1 \le i \le n} \left\{ p_{iik} - \sum_{j=1,\, j \ne i}^{n} |p_{ijk}| \right\} > 0, \qquad k = 1, 2, \ldots, l. \tag{17}$$

Assume that (2) *and* (5) *hold. If all solutions of the scalar difference equation*

$$u(t) - u(t - \sigma) + \sum_{k=1}^{l} q_k u(t - \tau_k) = 0 \tag{18}$$

are oscillatory, then all solutions of (1) *are also oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$. Let $y_i(t) = \delta_i x_i(t)$, $i = 1, 2, \ldots, n$. By Lemma 1, it follows from (1) that

$$y_i(t) - y_i(t - \sigma) + \sum_{k=1}^{l} \sum_{j=1}^{n} \bar{p}_{ijk} y_j(t - \tau_k) = 0, \qquad i = 1, 2, \ldots, n, \tag{19}$$

where

$$\bar{p}_{ijk} = \frac{\delta_i}{\delta_j} p_{ijk}, \qquad i, j = 1, 2, \ldots, n \quad \text{and} \quad k = 1, 2, \ldots, l,$$

and $y_i(t) > 0$ for $i = 1, 2, \ldots, n$. From (19), we have

$$y_i(t) - y_i(t-\sigma) + \sum_{k=1}^{l}\left(\bar{p}_{iik} y_i(t-\tau_k) + \sum_{j=1,\, j\neq i}^{n} \bar{p}_{ijk} y_j(t-\tau_k)\right) = 0, \quad i = 1, 2, \ldots, n \tag{20}$$

By summing (vertically) both sides of (20), we find that

$$v(t) - v(t-\sigma) + \sum_{k=1}^{l}\sum_{i=1}^{n}\left(\bar{p}_{iik} y_i(t-\tau_k) + \sum_{j=1,\, j\neq i}^{n} \bar{p}_{ijk} y_j(t-\tau_k)\right) = 0, \tag{21}$$

where $v(t) = \sum_{i=1}^{n} y_i(t)$. As for $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, l$, $|\bar{p}_{iik}| = p_{iik}$, and $|\bar{p}_{ijk}| = |p_{ijk}|$, it follows from (21) and (17) that

$$\begin{aligned} 0 &> v(t) - v(t-\sigma) + \sum_{k=1}^{l}\sum_{i=1}^{n}\left(p_{iik} y_i(t-\tau_k) - \sum_{j=1,\, j\neq i}^{n} |p_{ijk}| y_j(t-\tau_k)\right) \\ &\geq v(t) - v(t-\sigma) + \sum_{k=1}^{l} q_k v(t-\tau_k). \end{aligned} \tag{22}$$

It is easy to see that all the hypotheses of Lemma 2 are satisfied. Hence by (22) we see that the corresponding scalar difference equation (18) has an eventually positive solution $u(t)$. This contradicts the hypothesis that all solutions of (18) are oscillatory and completes the proof of the theorem. ∎

Remark 2. By Theorem 1, in order to find explicit sufficient conditions for the oscillation of all solutions of system (1), we only need to look for sufficient conditions for the oscillation of all solutions of the scalar difference equation (18). Thus by Theorem 3 in [3], we have the following result.

COROLLARY 1. *Assume that* (2) *holds and*

$$\sigma < \tau_k \quad \text{for } k = 1, 2, \ldots, l. \tag{23}$$

If

$$\sum_{k=1}^{l} q_k \left[\frac{\tau_k^{\tau_k}}{\sigma^{\sigma}(\tau_k - \sigma)^{\tau_k - \sigma}}\right]^{1/\sigma} > 1, \tag{24}$$

then all solutions of (1) *are oscillatory.*

4. GENERALIZATION

Theorem 1 in Section 2 can be generalized to the nonautonomous system of delay difference equations

$$x_i(t) - x_i(t-\sigma) + \sum_{k=1}^{l} \sum_{j=1}^{n} p_{ijk}(t) x_j(t-\tau_k) = 0, \qquad i = 1, 2, \ldots, n, \tag{25}$$

where

$$\sigma, \tau_k \in (0, \infty), \qquad p_{ijk} \in C\big[[t_0, \infty), R\big], \qquad i, j = 1, 2, \ldots, n \quad \text{and} \quad k = 1, 2, \ldots, l. \tag{26}$$

THEOREM 2. *Assume that* (26) *and* (5) *hold and*

$$q_k = \inf_{t \in [t_0, \infty)} \min_{1 \le i \le n} \left(p_{iik}(t) - \sum_{j=1,\, j \ne i}^{n} |p_{ijk}(t)| \right) > 0. \tag{27}$$

If all solutions of (18) *are oscillatory, then all solutions of* (25) *are also oscillatory.*

Proof. Suppose that (25) has a nonoscillatory solution $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$. Let $y_i(t) = \delta_i x_i(t) > 0$, $i = 1, 2, \ldots, n$. From (25) we have

$$y_i(t) - y_i(t-\sigma) + \sum_{k=1}^{l} \sum_{j=1}^{n} \bar{p}_{ijk}(t) y_j(t-\tau_k) = 0, \qquad i = 1, 2, \ldots, n, \tag{28}$$

where

$$\bar{p}_{ijk}(t) = \frac{\delta_i}{\delta_j} p_{ijk}(t), \qquad i, j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, l.$$

It follows from (28) that

$$\begin{aligned} y_i(t) &- y_i(t-\sigma) \\ &+ \sum_{k=1}^{l} \left(\bar{p}_{iik}(t) y_i(t-\tau_k) + \sum_{j=1,\, j \ne i}^{n} \bar{p}_{ijk}(t) y_j(t-\tau_k) \right) = 0, \\ & \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{29}$$

By summing both sides of (29) and using (27), we find that

$$
\begin{aligned}
0 &> v(t) - v(t-\sigma) \\
&\quad + \sum_{k=1}^{l} \sum_{i=1}^{n} \left(p_{iik}(t) y_i(t-\tau_k) - \sum_{j=1,\, j\neq i} |\bar{p}_{jik}(t)| y_i(t-\tau_k) \right) \\
&\geq v(t) - v(t-\sigma) + \sum_{k=1}^{l} q_k v(t-\tau_k),
\end{aligned}
$$

where $v(t) = \sum_{i=1}^{n} y_i(t)$. By Lemma 2, (18) has an eventually positive solution. This is a contradiction. The proof of Theorem 2 is complete. ∎

COROLLARY 2. *Assume that* (5), (23), (24), (26), *and* (27) *hold*. *Then all solutions of* (25) *are oscillatory*.

Remark 3. The results of this paper can be generalized further to nonlinear system of difference equations.

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